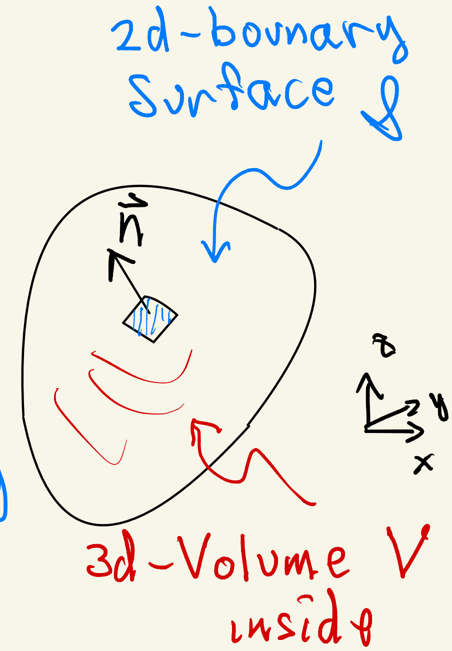
§ 16.4/5 Flux and Surface Integrals

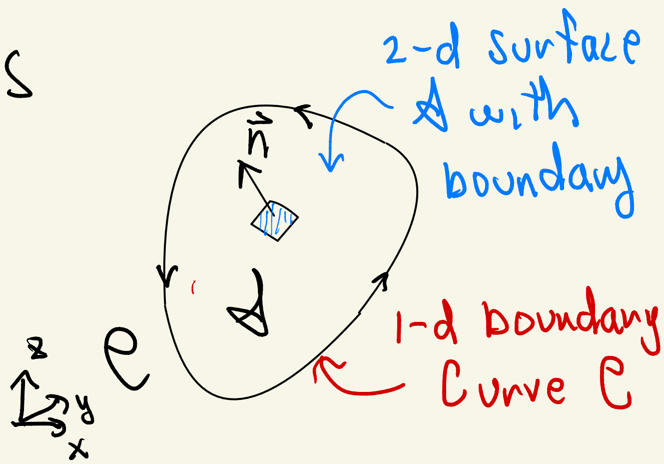
Recall: the Divergence Theorem:

$$\iiint_{V} \operatorname{Div}\vec{F}\, dv = \underbrace{\iint_{S} \vec{F}\cdot\vec{n}\, dS}_{\text{Flux of } \vec{F} \text{ thru the closed boundary } S}$$



Recall that Stokes Theorem reads

$$\underbrace{\iint_{S} \operatorname{Curl}\vec{F}\cdot\vec{n}\, dS}_{\text{Flux of Curl}\vec{F} \text{ thu } S} = \int_{C} \vec{F}\cdot\vec{T}\, ds$$

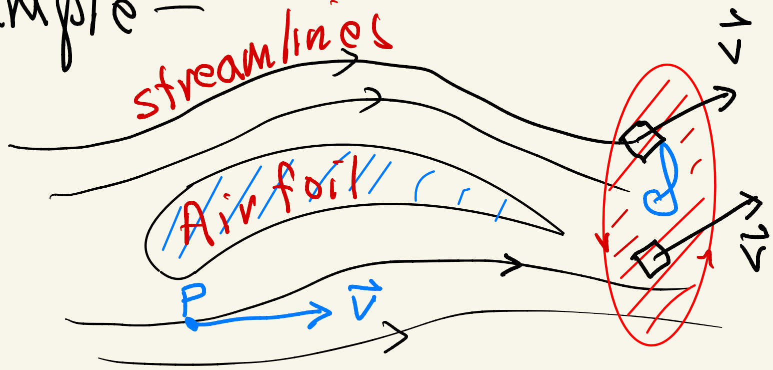


This reduces to Green's Thm in the special case when S reduces to a region R in xy-plane inside scc C in plane

Green's Thm: $$\iint_{R} \underbrace{N_x - M_y}_{\overrightarrow{\operatorname{Curl}(M,N,0)}\,\cdot\,\overrightarrow{(0,0,1)}}\, dA = \int_{C} \vec{F}\cdot\vec{T}\, ds$$

- **Flux** has a very important physical meaning:

Consider the fluid example –



A fluid particle P moves along curves $\vec{r}_p(t)$ called streamlines –

The fluid density $\delta(x,y,z) = \dfrac{\text{mass}}{\text{vol}}$ changes as the fluid "compresses" and "rarifies".

Q: Given a 2-d surface $\mathcal{S}$, what is the rate at which mass is passing thru $\mathcal{S}$?

Ans:
$$\text{Flux} = \iint_{\mathcal{S}} \vec{F}\cdot\vec{n}\, d\mathcal{S} = \frac{\text{Mass}}{\text{Time}} \text{ thru } \mathcal{S}$$

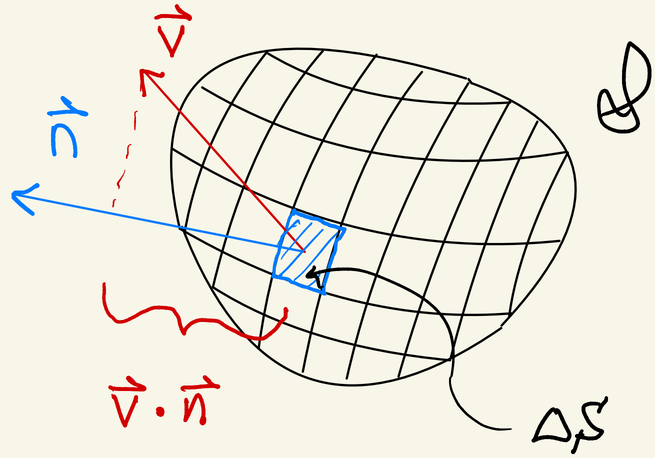
$\vec{F} = \delta\vec{v}$ = mass flux vector.

Check dimensions $[\delta\vec{v}] = \dfrac{\text{mass}}{\text{vol}}\,\dfrac{\text{dist}}{\text{time}} = \dfrac{M\,L}{L^3\,T} = \dfrac{M}{L^2\,T}$

$\dfrac{\text{mass}}{\text{area} \times \text{time}}$

<u>Conclude</u>: $\delta\vec{v} = \dfrac{\text{mass}}{\text{area}\times\text{time}}$ moving thru surface in direction $\vec{v}$

- Discretize $\mathcal{S}$ into a grid of small areas ΔS



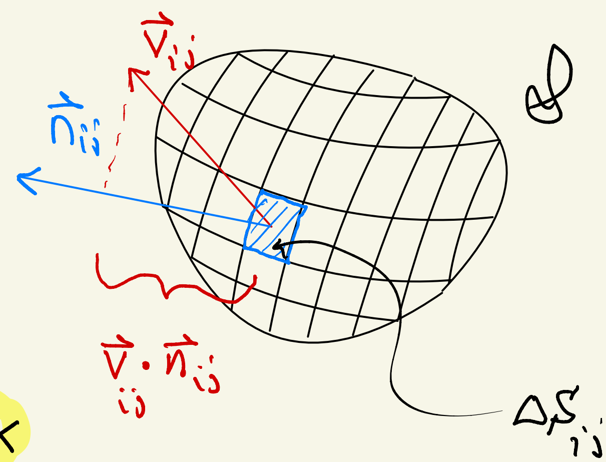
- Only the component of $\vec{v} \perp \Delta S$ contributes to mass flow thru ΔS. This is $\vec{v}\cdot\vec{n}$

- $\delta \vec{v}\cdot\vec{n} = \dfrac{\text{mass}}{\text{vol}}\ \dfrac{\text{dist}}{\text{time}}$ moving $\perp$ to ΔS

$$= \frac{\text{mass}}{\text{area time}} \text{ out thru } \Delta S$$

- $\delta \vec{v}\cdot\vec{n}\,\Delta S = \dfrac{\text{mass}}{\text{time}}$ out thru ΔS

- $\underbrace{\dfrac{\text{Total Mass}}{\text{Time}}}_{\text{out thru } \mathcal{S}} \simeq \displaystyle\sum_{ij} \delta_{ij} \vec{v}_{ij}\cdot\vec{n}_{ij}\,\Delta S_{ij}$



$$\frac{\text{Total Mass}}{\text{Time}} = \lim_{N\to\infty} \sum_{ij} \delta_{ij} \vec{v}_{ij}\cdot\vec{n}_{ij}\,\Delta S_{ij}$$

$$= \iint_{\mathcal{S}} \delta \vec{v}\cdot\vec{n}\, dS = \text{Flux}$$

- The argument works for any density –

$\delta = \frac{\text{charge}}{\text{vol}}, \frac{\text{contaminant}}{\text{vol}}, \frac{\text{"stuff"}}{\text{vol}}$

– assume its moving at velocity $\vec{v}$

– define $\delta\vec{v}$ = "stuff" flux vector

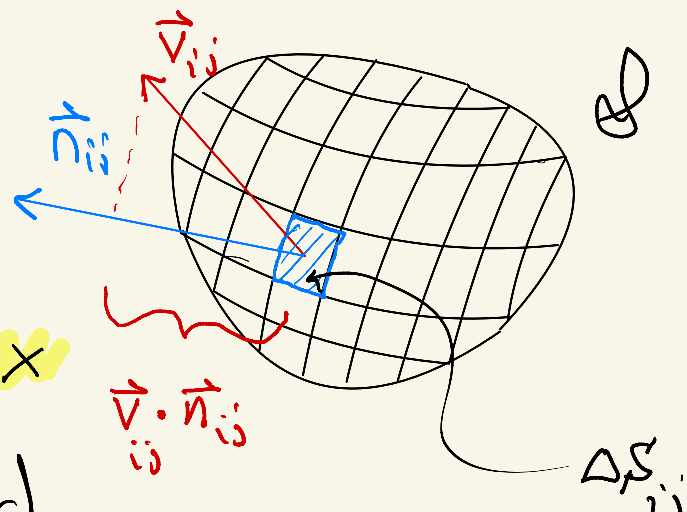
– then the "Flux thru $\mathcal{S}$" is defined as

$$\iint_{\mathcal{S}} \delta\vec{v}\cdot\vec{n}\,dS = \text{"rate at which stuff is passing thru } \mathcal{S} \text{ in direction of normal } \vec{n}\text{"}$$

Precisely –

$$\frac{\text{Total Stuff}}{\text{Time}} = \lim_{N\to\infty} \sum_{ij} \delta_{ij}\vec{v}_{ij}\cdot\vec{n}_{ij}\,\Delta S_{ij}$$

$$= \iint_{\mathcal{S}} \delta\vec{v}\cdot\vec{n}\,dS = \text{Flux}$$



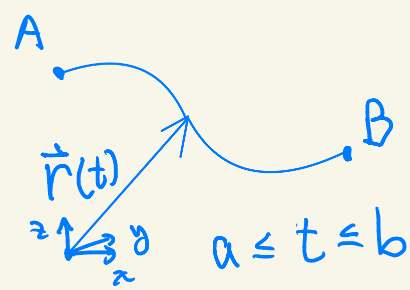
To calculate this we need to define the Riemann Sum in a Coordinate System!

Calculating surface integrals –

Turns out – there is a very natural way to define Flux integrals it terms of a coordinate system on the surface – analagous to coordinate systems on curves –

Defn: we call a coord system a parameterization

Recall: $\displaystyle\int_C \vec{F}\cdot\vec{T}\,ds = \int_a^b \vec{F}\cdot\vec{v}\,dt$

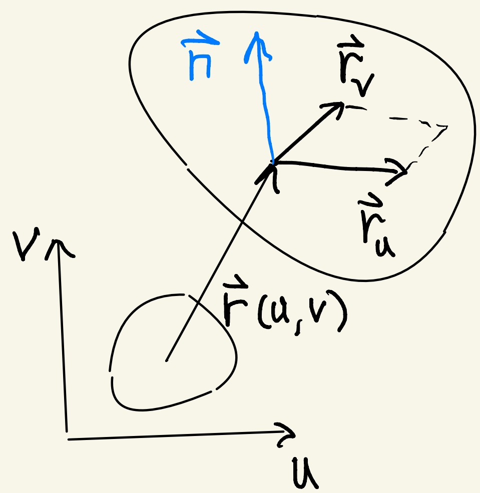


Similarly for surfaces –

$$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$$

$$\vec{r}_u = \langle x_u(u,v), y_u(u,v), z_u(u,v)\rangle$$

$$\vec{r}_v = \langle x_v(u,v), y_v(u,v), z_v(u,v)\rangle$$



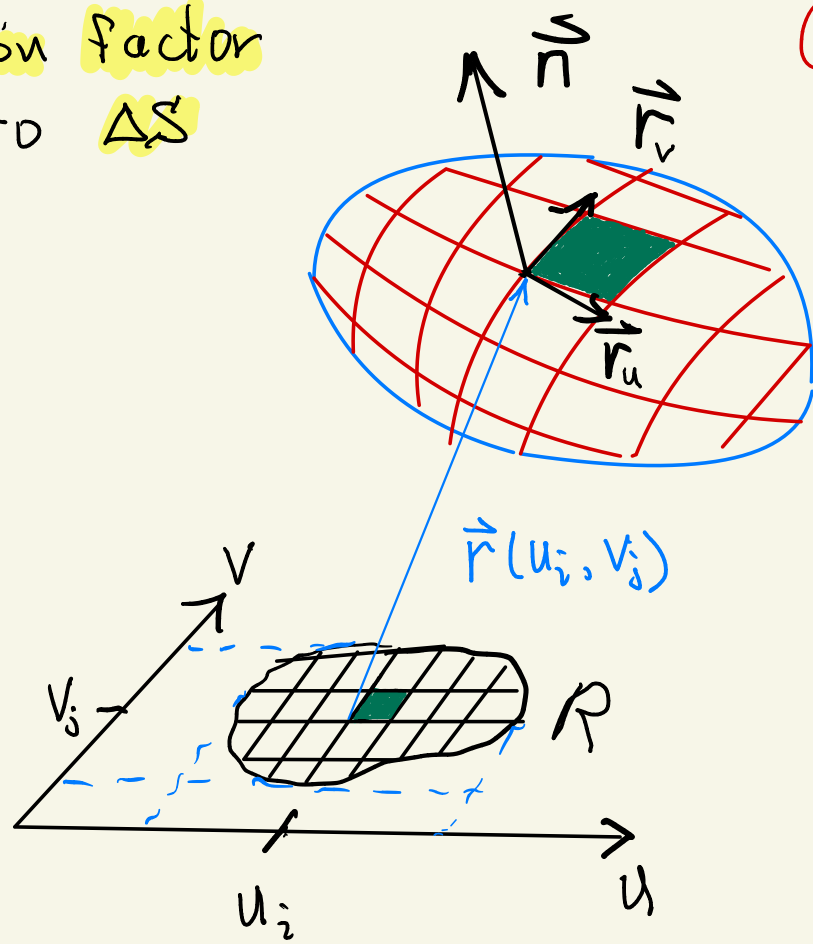
$$\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|} = \text{unit normal}$$

It remains to get the amplification factor –

Turns out: The amplification factor A that scales $\Delta u \Delta v$ to ΔS

$$\Delta S = A \, \Delta u \Delta v$$

is $A = |\vec{r}_u \times \vec{r}_v|$



From this fact we can compute a surface integral flux in a coordinate system:

$$\iint_S \vec{F} \cdot \vec{n} \, dS = \lim_{N \to \infty} \sum \vec{F}_{ij} \cdot \underbrace{\vec{n}_{ij}}_{\frac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}} \; \underbrace{\Delta S_{ij}}_{\|\vec{r}_u \times \vec{r}_v\| \Delta u \Delta v}$$

$$= \lim_{N \to \infty} \underbrace{\sum_{(u_i, v_j) \in R} \vec{F}_{ij} \cdot \frac{\vec{r}_u \times \vec{r}_v}{\cancel{\|\vec{r}_u \times \vec{r}_v\|}} \cancel{\|\vec{r}_u \times \vec{r}_v\|} \underbrace{\Delta u \Delta v}_{dA}}_{\text{Riemann Sum in } (u,v)}$$

$$= \iint_R \vec{F} \cdot (\vec{r}_u \times \vec{r}_v) \, dA$$

A Chapter 15 integral over a region R in uv-plane

Summary -

Line Integral: $\displaystyle\int_C \vec{F}\cdot\vec{T}\,ds = \int_a^b \vec{F}\cdot\vec{v}\,dt$

$\vec{r}(t): a \le t \le b$

Surface Integral: $\displaystyle\iint_S \vec{F}\cdot\vec{n}\,dS = \iint_R \vec{F}\cdot(\vec{r}_u \times \vec{r}_v)\,dA$

$\vec{r}(u,v): (u,v) \in R$

The only step that needs justification is the amplification factor -

Q: Why is $dS = |\vec{r}_u \times \vec{r}_v|\,du\,dv$?

We first recall the cross-product.

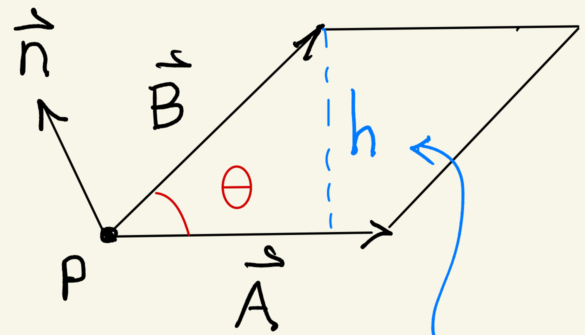
Cross Product: $\vec{A} = \overrightarrow{(a_1, a_2, a_3)}$ $\vec{B} = \overrightarrow{(b_1, b_2, b_3)}$

How to calculate it:

$$\vec{A} \times \vec{B} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2 b_3 - a_3 b_2)\underset{\sim}{i} - (a_1 b_3 - a_3 b_1)\underset{\sim}{j} + (a_1 b_2 - a_2 b_1)\underset{\sim}{k}$$

Geometric meaning:

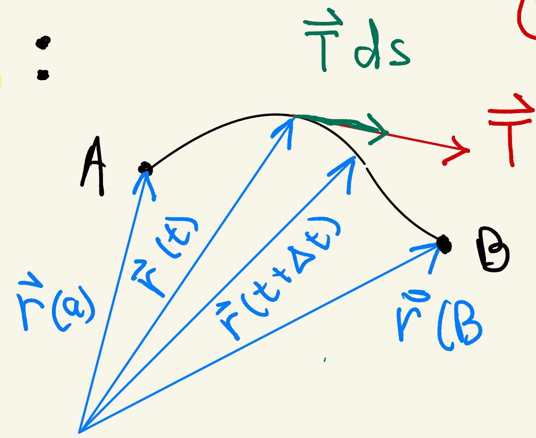
$$\vec{A} \times \vec{B} = \underbrace{\|\vec{A}\| \underbrace{\|\vec{B}\| \sin\theta}_{h}}_{\text{Area of the parallelogram}} \ \vec{n}$$



parallelogram determined by $\vec{A}$ and $\vec{B}$

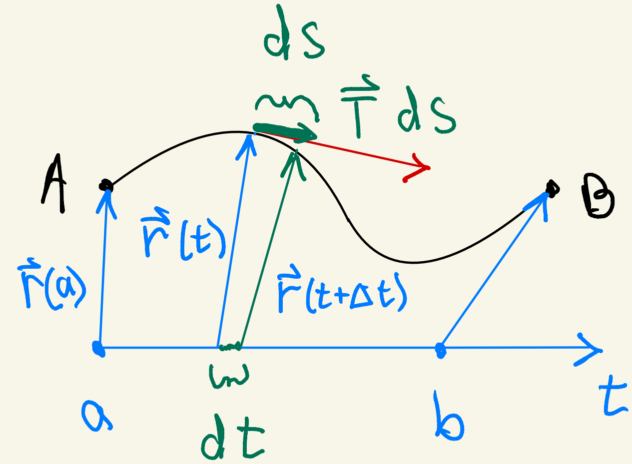
Conclude: The cross product points in direction $\perp$ $\vec{A}$ & $\vec{B}$ (direction by right hand rule) and has a length = area of parallelogram.

- Now consider a curve $\mathcal{C}$:



- At a point $\vec{r}(t)$ on a curve, the vector $\vec{T}\,ds$ points tangent to $\mathcal{C}$, and has a length $ds = \vec{r}\,'(t)\,dt$

(ds is distance along tangent line, a good approximation to Δs along curve, when ds is <u>small</u>)



- <u>Similarly</u>: Given a surface $\vec{r}(u,v)$

$\vec{r}_u\,du = \vec{T}_u\,ds$ is the vector on the side of ΔS tangent to curve $\vec{r}_v(u)$

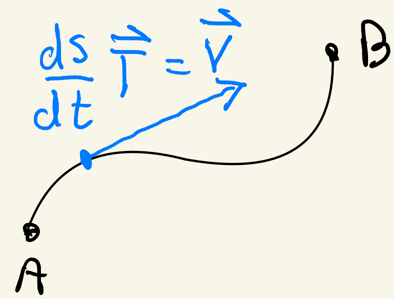
$\vec{r}_v\,dv = \vec{T}_v\,ds$ is the vector on the side of ΔS tangent to curve $\vec{r}_u(v)$

Explanation: Why $|\vec{r}_u \times \vec{r}_v|$ is the amplification factor for area I.e.,

$$dS = |\vec{r}_u \times \vec{r}_v| \, du \, dv$$

(1) First recall that for a curve $\vec{r}(t)$, we have $\|\vec{v}\| = \|\vec{r}\,'(t)\| = \dfrac{ds}{dt} \Rightarrow ds = \|\vec{r}\,'(t)\| \, dt$

Thus $\vec{T} ds$ is a vector of length ds pointing tangent



(2) Similarly, $\vec{r}(u,v)$ parameterizes a surface — and at fixed v

$\vec{r}(u,v) = \vec{r}_v(u)$ is a curve with parameter u

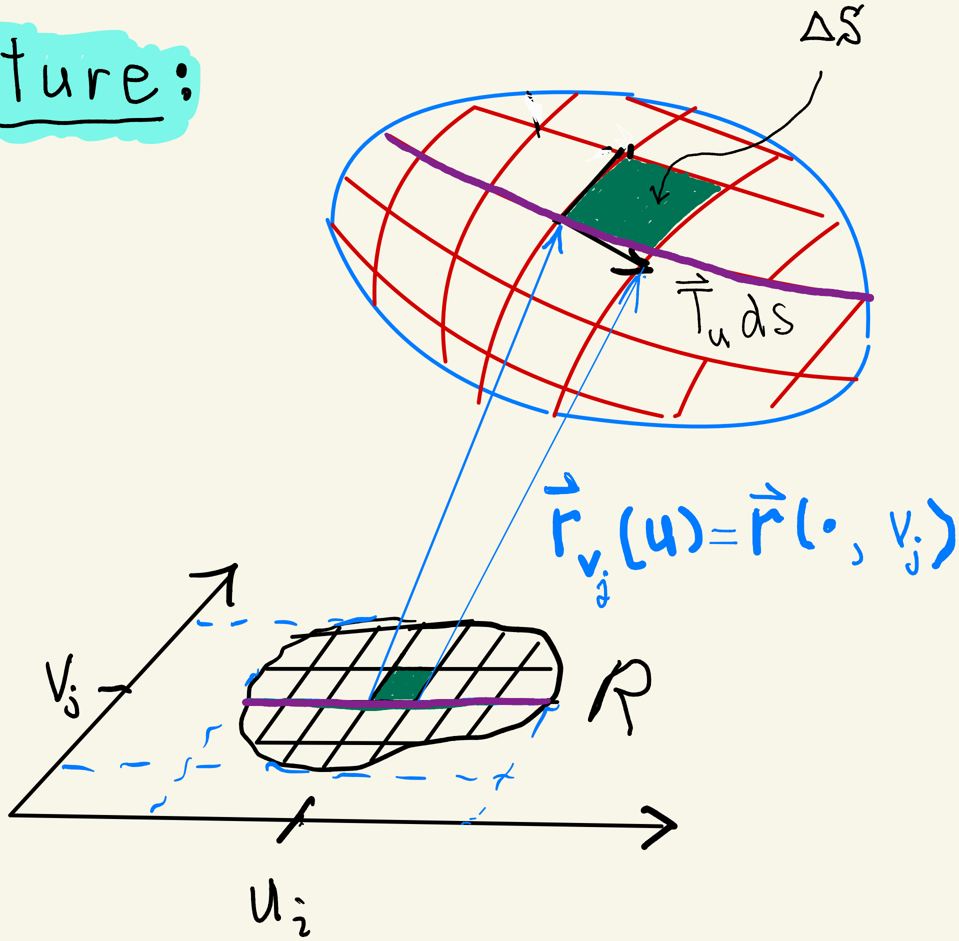
$\dfrac{\partial \vec{r}}{du}(u,v) = \dfrac{d\vec{r}_v}{du} = \vec{T}_u \, ds$ is one side of ∥-ogram ΔS

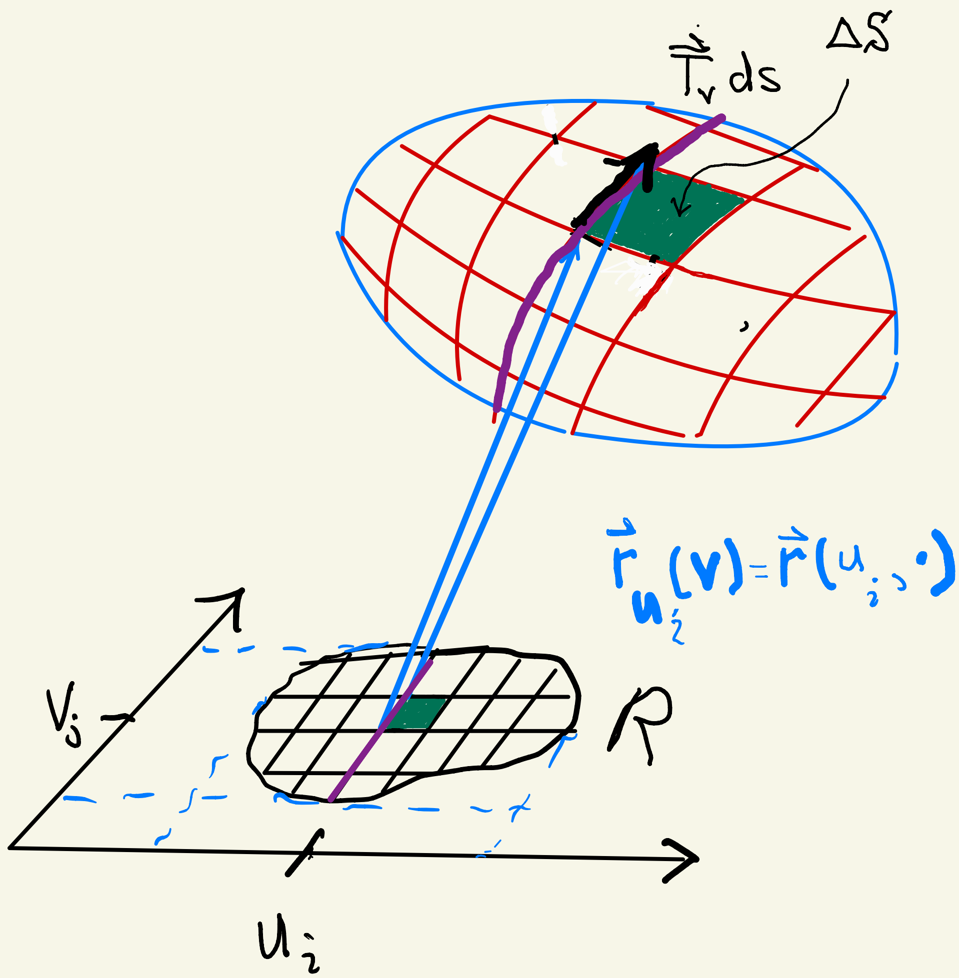
Same for v:

$\vec{r}(u,v) = \vec{r}_u(v)$ is a curve with parameter v

$\dfrac{\partial \vec{r}}{\partial v}(\,,v) = \dfrac{d\vec{r}_u}{dv} = \vec{T}_v \, ds$ is the other side of ∥-ogram ΔS

Picture:


ΔS
$\vec{T}_u\, ds$
$\vec{r}_{v_j}(u) = \vec{r}(\cdot, v_j)$
v_j
R
u_i


ΔS
$\vec{T}_v\, ds$
$\vec{r}_{u_i}(v) = \vec{r}(u_i, \cdot)$
v_j
R
u_i

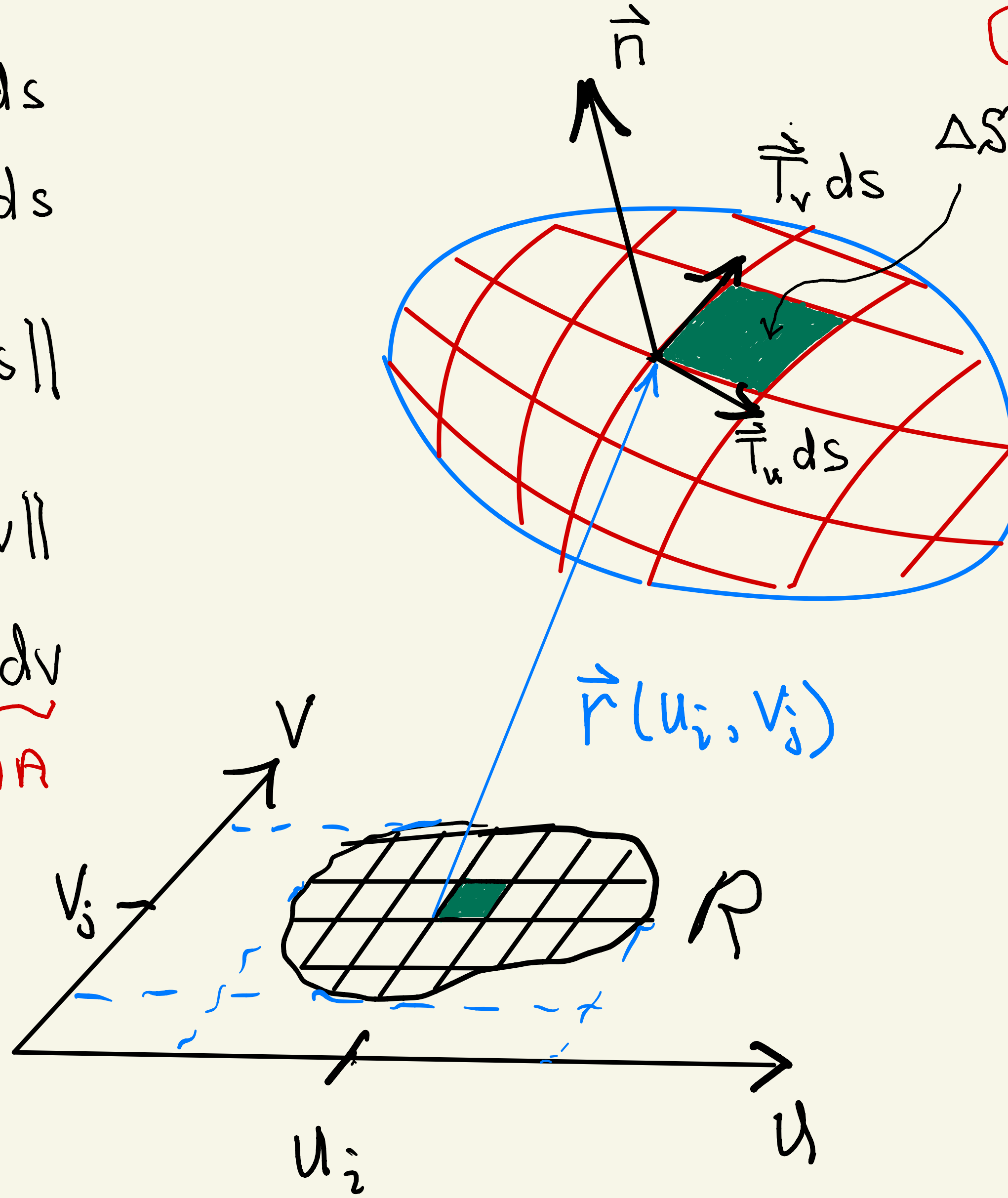
(3) $\vec{r}_u \, du = \vec{T}_u \, ds$

$\vec{r}_v \, dv = \vec{T}_v \, ds$

$$dS = \| \vec{T}_u ds \times \vec{T}_v ds \|$$

$$= \| \vec{r}_u du \times \vec{r}_v dv \|$$

$$= \underbrace{\| \vec{r}_u \times \vec{r}_v \|}_{\text{Amplification Factor for Area}} \underbrace{du\,dv}_{dA}$$



Summary: $dS = \| \vec{r}_u \times \vec{r}_v \| \underbrace{du\,dv}_{dA}$

so

$$\iint_{S} \vec{F} \cdot \vec{n} \, dS = \iint_{R_{uv}} \vec{F} \cdot \underbrace{\frac{\vec{r}_u \times \vec{r}_v}{\cancel{\| \vec{r}_u \times \vec{r}_v \|}}}_{\vec{n}} \underbrace{\cancel{\| \vec{r}_u \times \vec{r}_v \|} dA}_{dS}$$

$$= \iint_{R_{uv}} \vec{F} \cdot \vec{r}_u \times \vec{r}_v \, dA$$

Example: Assume a surface S is given by

$$\vec{r}(u,v) = \langle \underbrace{u-v}_{x(u,v)}, \underbrace{u+v}_{y(u,v)}, \underbrace{u+2v}_{z(u,v)} \rangle \qquad \begin{array}{l} 0 \le u \le 1 \\ 0 \le v \le 2 \end{array}$$

Assume a density $\delta(x,y,z) = x \ \frac{kg}{m^3}$ is moving thru the surface at velocity $\vec{v} = y\langle 1,2,1\rangle \ \frac{m}{s}$

Find the rate and direction at which mass is passing thru the surface.

Soln: $\vec{F} = \delta \vec{v} = xy\langle 1,2,1\rangle$ is the mass flux vector

Flux $= \iint_S \vec{F}\cdot\vec{n}\,dS = \frac{\text{mass}}{\text{time}}$ thru S in direction of normal vector $\vec{n}$.

$$\vec{r}_u = \frac{\partial \vec{r}}{\partial u} = \frac{\partial}{\partial u}\langle u-v, u+v, u+2v\rangle = \langle 1,1,1\rangle$$

$$\vec{r}_v = \frac{\partial \vec{r}}{\partial v} = \frac{\partial}{\partial v}\langle u-v, u+v, u+2v\rangle = \langle -1,1,2\rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 1 & 1 & 1 \\ -1 & 1 & 2 \end{vmatrix} = \underset{\sim}{i}(2-1) - \underset{\sim}{j}(2-(-1)) + \underset{\sim}{k}(1-(-1))$$

$$= \underset{\sim}{i} - 3\underset{\sim}{j} + 2\underset{\sim}{k} = \overrightarrow{(1,-3,2)}$$

$$\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|} = \frac{\overrightarrow{(1,-3,2)}}{\sqrt{1^2+(-3)^2+2^2}} = \frac{1}{\sqrt{14}}\,\overrightarrow{(1,-3,\underset{\substack{\uparrow \\ \text{positive}}}{2})}$$

Since z-component of $\vec{n} > 0$, this is upward normal

$$\iint_S \vec{F} \cdot \vec{n}\, dS = \iint_{R_{uv}} \underbrace{xy\,\overrightarrow{(1,2,1)}}_{\vec{F}} \underbrace{\frac{1}{\cancel{\sqrt{14}}}\,\overrightarrow{(1,-3,2)}}_{\vec{n}} \underbrace{\cancel{\sqrt{14}}\, dA}_{dS}$$

$x = u - v,\ y = u + v$

$$= \int_0^2 \int_0^1 (u-v)(u+v)(1-6+2)\, du\, dv \qquad \equiv \iint_{R_{uv}} \vec{F} \cdot \vec{r}_u \times \vec{r}_v\, dA$$

$$= -3\int_0^2 \int_0^1 u^2 - v^2\, du\, dv = -3\int_0^2 \left.\frac{u^3}{3} - v^2 u\right]_{u=0}^{u=1} dv = -3\int_0^2 \frac{1}{3} - v^2\, dv$$

$$= -3\left[\frac{1}{3}v - \frac{v^3}{3}\right]_{v=0}^{v=2} = -3\left[\left(\frac{2}{3} - \frac{8}{3}\right) - 0\right] = 6\ \frac{kg}{s} > 0$$

$\Rightarrow$ upward thru S ✓